Tangents and Normals

1. GEOMETRICAL INTERPRETATION OF THE DERIVATIVE

Let $y = f(x)$ be a given function. Its derivative $f'(x)$ or $\frac{dy}{dx}$ is equal to the trigonometrical tangent of the angle which tangent to the graph of the function at the point (x, y) makes with the positive direction of x-axis. Therefore $\frac{dy}{dx}$ is the slope of the tangent.

Thus $\boxed{f'(x) = \frac{dy}{dx} = \tan\Psi}$

Hence at any point of a curve $y = f(x)$

(i) Inclination of the tangent (with x-axis) = $\tan^{-1}\left(\frac{dy}{dx}\right)$

(ii) Slope of the tangent = $\frac{dy}{dx}$

(iii) Slope of the normal = $-\frac{1}{\left(\frac{dy}{dx}\right)} = -\left(\frac{dx}{dy}\right)$

(iv) Slope of the tangent at (x_1, y_1) is denoted by $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

(v) Slope of the normal at (x_1, y_1) is denoted by $\left(-\frac{dx}{dy}\right)_{(x_1, y_1)}$

2. EQUATION OF TANGENT

(i) The equation of tangent to the curve $y = f(x)$ at (x_1, y_1) is $(Y - y_1) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}(X - x_1)$

Slope of tangent $= \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

(ii) If a tangent is parallel to the axis of x then $\Psi = 0$

$$\therefore \frac{dy}{dx} = \tan\Psi = \tan 0 = 0 \Rightarrow \boxed{\frac{dy}{dx} = 0}$$

(iii) If the equation of the curve be given in the parametric form say $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right) = \frac{g'(t)}{f'(t)}$$

The equation of tangent at any point 't' on the curve is given by $\boxed{y - g(t) = \frac{g'(t)}{f'(t)}(x - f(t))}$

(v) If the tangent is perpendicular to the axis of x, then $\Psi = \frac{\pi}{2}$

$$\therefore \frac{dy}{dx} = \tan\Psi = \tan\frac{\pi}{2} = \infty \Rightarrow \boxed{\frac{dx}{dy} = 0}$$

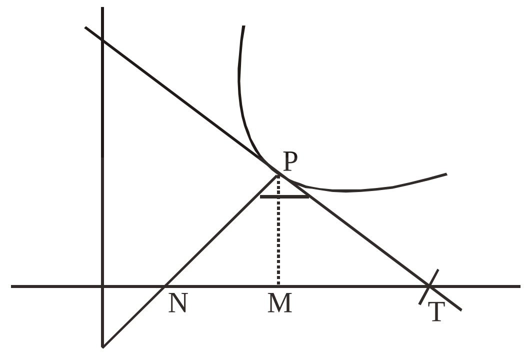
(vi) $\because$ The value of Ψ always lie in $(-\pi, \pi]$

(vii) If the tangent at any point on the curve is equally inclined to both the axes then $\left(\frac{dy}{dx}\right) = \pm 1$

4. LENGTH OF THE TANGENT

$$PT = y\,\mathrm{cosec}\,\Psi = y\sqrt{1+\cot^2\Psi}$$

$$= \left| y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}} \right| \quad \text{or} \quad \left| \frac{y\sqrt{1+(dy/dx)^2}}{(dy/dx)} \right|$$



5. LENGTH OF SUB-TANGENT

$$TM = y\cot\Psi = \frac{y}{\tan\Psi} = \left| y\frac{dx}{dy} \right| \quad \text{or} \quad \frac{y}{(dy/dx)}$$

6. EQUATION OF NORMAL

The equation of normal

(i) at the point $P(x_1, y_1)$ on the curve $Y = f(x)$ is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x - x_1)$$

$$\boxed{y - y_1 = -\left(\frac{dx}{dy}\right)_{(x_1,y_1)}(x - x_1)}$$

Slope of normal = $-\frac{1}{\text{slope of tangent}} = -\frac{1}{\left(\frac{dy}{dx}\right)}$

(ii) If the normal is parallel to the axis of y, then $\Rightarrow \Psi = 0$

$$\therefore \frac{dy}{dx} = \tan\Psi = \tan 0 = 0$$

(iii) If the normal is parallel to the axis of x, then $\therefore \frac{dx}{dy} = 0$

7. LENGTH OF NORMAL

$$PN = y\sec\Psi \quad = y\sqrt{1+\tan^2\Psi} = \left|y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\right|$$

8. LENGTH OF SUB-NORMAL

$$MN = y\tan\Psi = \left|y\frac{dy}{dx}\right|$$

9. ANGLE OF INTERSECTION OF TWO CURVES

If two curves $y = f_1(x)$ and $y = f_2(x)$ intersect at a point p, the angle between their tangents at p is defined as the angle between these two curves at p. But slopes of tangents at P are $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{dy}{dx}\right)_2$

So at p their angle of intersection ψ is given by

$$\tan\psi = \left|\frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1+\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2}\right| \quad \text{or} \quad \boxed{\tan\psi = \pm\frac{m_1 - m_2}{1+m_1m_2}}$$

1. If two curves cut perpendicular then $\psi = \frac{\pi}{2}$

   $\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2 = -1$ or $m_1m_2 = -1$

2. If two curves are parallel $\psi = 0^0$

   $\left(\frac{dy}{dx}\right)_1 = \left(\frac{dy}{dx}\right)_2$ or $m_1 = m_2$

10. ROLLE'S THEOREM

If a function $f(x)$ is defined on $[a,b]$ satisfying

(i) f is continuous on $[a,b]$

(ii) f is differentiable on (a,b)

(iii) $f(a) = f(b)$ then there exists $c \in (a,b)$; Such that $f'(c) = 0$

11. LANGRAGE'S MEAN VALUE THEOREM

If a function $f(x)$ is defined on $[a,b]$ satisfying

(i) f is continuous on $[a,b]$

(ii) f is differentiable on (a,b) then there exists $c \in (a,b)$ Such that $f'(c) = \frac{f(b)-f(a)}{b-a}$